



Shifted-Antimagic Labelings for Graphs

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Abstract

The concept of antimagic labelings of a graph is to produce distinct vertex sums by labeling edges through consecutive numbers starting from one. A long-standing conjecture is that every connected graph, except a single edge, is antimagic. Some graphs are known to be antimagic, but little has been known about sparse graphs, not even trees. This paper studies a weak version called k -shifted-antimagic labelings which allow the consecutive numbers starting from $k + 1$, instead of starting from 1, where k can be any integer. This paper establishes connections among various concepts proposed in the literature of antimagic labelings and extends previous results in three aspects:

- Some classes of graphs, including trees and graphs whose vertices are of odd degrees, which have not been verified to be antimagic are shown to be k -shifted-antimagic for sufficiently large k .
- Some graphs are proved k -shifted-antimagic for all k , while some are proved not for some particular k .
- Disconnected graphs are also considered.

Keywords Antimagic labeling · Disconnected graphs · Trees

1 Introduction

Graph labeling problems are interesting and broadly studied. The concept of *antimagic labelings* was first introduced by Hartsfield and Ringel [6]. Here is the definition.

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Definition 1.1 Let $G = (V, E)$ be a graph with $|E(G)| = m$, and let f be an injective function f from $E(G)$ to some subset $S \subset \mathbb{R}$. The vertex sum of a vertex v , induced by f , is defined to be $\phi_f(v) := \sum_{e \sim v} f(e)$, where $e \sim v$ means the edge e is incident to v . If $S = \{1, 2, \dots, m\}$ and the vertex sums are all distinct for all vertices, then we say G is antimagic and f is an antimagic labeling of G .

It is trivial that this kind of labelings does not exist if $G = K_2$. Hartsfield and Ringel [6] proved that some graphs are antimagic, including the paths P_n , the cycles C_n , and the complete graphs K_n for $n \geq 3$, and came up with the following two conjectures.

Conjecture 1.1 [6] *Every connected graph with at least three vertices is antimagic.*

Conjecture 1.2 [6] *Every tree other than K_2 is antimagic.*

The former conjecture has been proved to be true for graphs with many edges or graphs whose vertex degrees are equal. In [1], Alon, Kaplan, Lev, Roditty, and Yuster proved that dense graphs are antimagic. Precisely, they proved that a graph G with minimum degree $\delta(G) \geq c \log |V|$ for some constant c , or with maximum degree $\Delta(G) \geq |V(G)| - 2$ is antimagic. The antimagic labeling of regular graphs has been investigated by many groups of researchers [4, 5, 10, 12, 17, 18], and is completely solved recently by Chang, Liang, Pan, and Zhu [2].

Generally speaking, in contrast to dense graphs, little has been known about the antimagic labeling of sparse graphs. The most celebrated result on trees is given by Kaplan, Lev, and Roditty [9] (with a small error in the proof corrected by Liang, Wong, and Zhu [13]) that every tree with at most one vertex of degree two is antimagic. Other classes of trees that satisfy Conjecture 1.2 but not mentioned previously include the spider graphs [8, 14], the double spider graphs [3], and the caterpillars with some extra assumptions [11].

Wang and Hsiao [18] introduced an analogue concept, called k -antimagic labelings, in the context of antimagic labelings of the Cartesian product and the lexicographic product of sparse graphs. In a k -antimagic labeling, they label the edges with $k + 1$ through $k + |E(G)|$ for a nonnegative integer k rather than with 1 through $|E(G)|$, and still ask for distinct vertex sums. By translating the labels of edges of an antimagic graph, they were able to produce antimagic labelings of the Cartesian product of some graphs. Notice that only nonnegative integers k are considered in their paper. We also make a remark about a different k -antimagic labeling, by Hefetz [7] and Wong and Zhu [19], which allows k more labels $|E(G)| + 1, \dots, |E(G)| + k$, i.e., the labels are starting from 1 through $|E(G)| + k$.

This paper follows the definition of k -antimagic labelings in [18] and generalizes it to a broader setting where k can be any integer. To avoid confusion between the k -antimagic labelings in [7, 19] and in [18], the labelings considered in the paper will be called k -shifted-antimagic labelings.

Definition 1.2 Let G be a graph with $|E(G)| = m$. Given $k \in \mathbb{Z}$, if there exists an injective function f from $E(G)$ to $\{k + 1, k + 2, \dots, k + m\}$ such that the vertex sums $\phi_f(v)$ are all distinct for all vertices $v \in V(G)$, then we say G is k -shifted-antimagic and f is a k -shifted-antimagic labeling of G .

Regardless the original purpose in [18], the study of k -shifted-antimagic labelings of graphs is interesting in its own. Obviously, it is a natural generalization of the traditional antimagic labelings. Indeed, an antimagic graph is 0-shifted-antimagic. Now that the two conjectures by Hartsfield and Ringel are still far from resolved, one may naturally wonder if they can be proved k -shifted-antimagic for some k . This paper gives an affirmative answer to the k -shifted-antimagic version for some classes of graphs that have not yet been proven antimagic, including trees and graphs whose vertex degrees are odd.

An interesting feature of k -shifted-antimagic labelings is that simply shifting the used labels by one in a k -shifted-antimagic labeling does not guarantee a $(k + 1)$ -shifted-antimagic labeling. Interestingly, if a graph G is k -shifted-antimagic for some k , then it can be shown that there exists an integer $k(G)$ such that G is k' -shifted-antimagic for any $|k'| \geq k(G)$.

A naturally raised question is to characterize the spectrum of integers k of a given graph such that it is k -shifted-antimagic. On the one hand, some examples are fully characterized in this paper. However, the provided examples are only small graphs due to the difficulty of determining a graph not k -shifted-antimagic for a certain k . On the other hand, there exist some graphs that are k -shifted-antimagic for all k . We call such a graph *absolutely antimagic*. Wang and Hsiao noticed that a graph G is k -shifted-antimagic for all $k \geq 0$ if there is an antimagic labeling f of G such that $\phi_f(u) > \phi_f(v)$ whenever $\deg(u) > \deg(v)$. Such a labeling f and graph G were called *strongly antimagic* later in [8]. Obviously, regular graphs are strongly antimagic and in fact they are absolutely antimagic by a straightforward argument. This paper provides a non-regular example by proving that the paths P_n , $n \geq 6$, are absolutely antimagic.

Disconnected graphs are also taken into account in this paper, although they have been received little attention in the literature. One reason could be the existence of disconnected graphs that are not antimagic, and proving not antimagic for a graph is relatively difficult because it needs to go through every possible labeling. Shang et al. [16] first pointed out the fact that a graph is barely antimagic if it contains too many components isomorphic to P_3 . Precisely, they proved that the union of a star S_n and c copies of P_3 is 0-shifted-antimagic if and only if $c \leq \min\{2n + 1, \frac{2n-5+\sqrt{8n^2-24n+17}}{2}\}$. Shang [15] proved that the graph consisting of c copies of P_3 is antimagic if and only if $c = 1$. We extend their results by showing that for any graph G there exists an integer c such that the union of G and c copies of P_3 is not antimagic, and providing a necessary and sufficient condition for the graph of c copies of P_3 is k -shifted-antimagic. In addition, we demonstrate that the graphs $2P_4$ and $2S_3$ are not k -shifted-antimagic if and only if $k = -2$ and $k = -5$. These are the only examples found so far that the integers k for which the graph is not k -shifted-antimagic do not appear consecutively.

The rest of the paper is organized as follows. In Sect. 2, we present results on the regular graphs, the forests, and the graphs without vertices of even degrees. Section 3 demonstrates some graphs that are not k -shifted-antimagic, and characterizes the spectrum of values of k for which a given graph is not k -shifted-antimagic. These graphs include the trees of diameter at most four and some

disconnected graphs. We present some remarks and open problems in the last section.

2 The k -Shifted-Antimagic Labeling

This section starts with a lemma due to a simple observation that after increasing all labels by 1 in a labeling of a graph, the more degree a vertex has the more increments it gets.

Lemma 2.1 *Given a graph G , if there exists an injective function f from $E(G)$ to $\{1, 2, \dots, m\}$ such that $\phi_f(v) \neq \phi_f(u)$ whenever $\deg(v) = \deg(u)$ for distinct vertices u and v , then G is k -shifted-antimagic for any sufficiently large k .*

Proof Let f be a function satisfying the required condition. Consider $f' = f + k$, where $k \geq (m-1)(\Delta(G)-1)$ and $\Delta(G)$ is the maximum degree of G . For two vertices u and v with $\deg(u) < \deg(v)$, we have

$$\begin{aligned} \phi_{f'}(v) - \phi_{f'}(u) &= [k\deg(v) + \phi_f(v)] - [k\deg(u) + \phi_f(u)] \\ &\geq [(m-1)(\Delta(G)-1)\deg(v) + \phi_f(v)] - [(m-1)(\Delta(G)-1)\deg(u) + \phi_f(u)] \\ &\geq [(m-1)(\Delta(G)-1)(\deg(v) - \deg(u)) - (m-1)\deg(u)] + (\deg(v) - \deg(u)) \\ &> 0. \end{aligned}$$

Clearly, $\phi_{f'}(v) \neq \phi_{f'}(u)$ if $\deg(v) = \deg(u)$ for distinct vertices u and v . Hence, f' is a k -shifted-antimagic labeling. \square

We call a labeling with the property described in Lemma 2.1 an *SDDS-labeling* (same-degree distinct-sum). Although it is unclear about the existence of the SDDS-labeling for general graphs, we have a method to construct it for various graphs. The following “level-by-level labeling algorithm” is originally used by Cranston, Liang, and Zhu [5] to find the antimagic labelings of odd regular graphs. Their ideas are sketched briefly in the following. First, pick a vertex w from the given graph G . Partition $V(G)$ into levels L_0, L_1, \dots, L_d , where $L_i = \{u \mid d(w, u) = i\}$ and d is the furthest distance of a vertex from w . Let $G[L_i]$ and $G[L_i, L_{i-1}]$ be the subgraph induced by L_i and the bipartite subgraph induced by the two parts L_i and L_{i-1} , respectively. Then construct a labeling f by labeling the edges in $G[L_d]$, $G[L_d, L_{d-1}]$, \dots , $G[L_1]$ and $G[L_1, L_0]$ in order using the smallest unused labels with some additional rules. We use this method to get the following two theorems.

Theorem 2.2 *If G is a forest without a component isomorphic to K_2 , then G is k -shifted-antimagic for sufficiently large k .*

Proof We prove this theorem by showing that every forest admits an SDDS-labeling. It suffices to prove the statement for G being a tree since if we label the components by nonoverlapping intervals of integers, then two vertices of the same degree in different components must have distinct vertex sums.

Let G be a tree. Then $E(G[L_i])$ is empty for each i , and we only need to label the edges in $G[L_i, L_{i-1}]$. Let $|E(G[L_i, L_{i-1}])| = m_i$ for $i = d, d-1, \dots, 1$. Now we define a labeling f on $E(G)$ as follows. First, label the edges in $G[L_d, L_{d-1}]$ arbitrarily with $1, \dots, m_d$. Suppose that all the edges in $G[L_d, L_{d-1}] \cup \dots \cup G[L_i, L_{i-1}]$ for some $i \leq d$ are labeled. For each vertex $v \in L_{i-1}$, there exists a unique edge $e_v \in G[L_{i-1}, L_{i-2}]$ incident to v . Define the partial vertex sum of v by

$$\phi'_f(v) := \sum_{e \sim v, e \neq e_v} f(e).$$

We may assume $\phi'_f(v_1) \leq \phi'_f(v_2) \leq \dots \leq \phi'_f(v_{m_{i-1}})$ for vertices $v_1, v_2, \dots, v_{m_{i-1}} \in L_{i-1}$. Finally, label e_{v_1}, e_{v_2}, \dots with $1 + \sum_{l=i}^d m_l, 2 + \sum_{l=i}^d m_l, \dots, \sum_{l=i-1}^d m_l$, accordingly.

Now we show that if two vertices u and v have the same degree, then $\phi_f(u) \neq \phi_f(v)$. The labeling rule for e_u and e_v promises that $\phi_f(u) \neq \phi_f(v)$ for any u and v in the same L_i . Consider $u \in L_i, v \in L_j, i < j$, and $\deg(v) = \deg(u)$. If $\deg(u) = \deg(v) = 1$, then $\phi_f(u) \neq \phi_f(v)$ as v and u are not adjacent for otherwise $G = K_2$. Suppose $\deg(u) = \deg(v) \geq 2$. If $i = 0$, then u is the root w , which is adjacent to the largest $\deg(u)$ labels. Thus, $\phi_f(u) > \phi_f(v)$ follows. Assume that $0 < i < j$. Observe that one of the edges incident to u is in $G[L_i, L_{i-1}]$ and others are in $G[L_{i+1}, L_i]$. Similarly, one of the edges incident to v is in $G[L_j, L_{j-1}]$ and others are in $G[L_{j+1}, L_j]$. Since all the labels of edges in $G[L_i, L_{i-1}]$ ($G[L_{i+1}, L_i]$) are greater than all the labels of edges in $G[L_j, L_{j-1}]$ ($G[L_{j+1}, L_j]$), one can conclude that $\phi_f(u) > \phi_f(v)$. Hence, f is an SDDS-labeling. \square

Theorem 2.3 *If a graph G consists of vertices of odd degrees and contains no component isomorphic to K_2 , then G is k -shifted-antimagic for sufficiently large k .*

Proof Similarly, it suffices to prove the statement for connected graphs. We first pick a vertex w in G and define L_i 's, $G[L_i]$'s, and $G[L_i, L_{i-1}]$'s accordingly as before. For each bipartite graph $G[L_i, L_{i-1}]$, we shall look for an injection σ_i from L_i to $E(G[L_i, L_{i-1}])$ via $\sigma_i(v) = vu \in E(G)$, with $v \in L_i$ and $u \in L_{i-1}$, which satisfies the following properties:

1. The graph $G[L_i, L_{i-1}] - \sigma_i(L_i)$ can be decomposed into edge-disjoint open trails T_i 's, where an open trail T_i is a sequence of vertices $v_1 v_2 \dots v_l$ for some l , $v_i v_{i+1} \in E(G)$, $v_i v_{i+1} \neq v_j v_{j+1}$ for $i \neq j$, and the initial vertex v_1 is not the same as the terminal vertex v_l .
2. No two trails share the initial or terminal vertices. In other words, if $T_1 = v_1 v_2 \dots v_a$ and $T_2 = u_1 u_2 \dots u_b$, then $\{v_1, v_a\} \cap \{u_1, u_b\} = \emptyset$.

Indeed, the existence of such an injection has been proved by Cranston, Liang, and Zhu [5] in their “Helpful Lemma”.

Suppose that the injections σ_i 's for all L_i 's have done. Let $\sigma_i(L_i) = \{\sigma_i(u) \mid u \in L_i\}$. We construct a labeling f by assigning the smallest unused labels to the edges in $G[L_d]$, $G[L_d, L_{d-1}] - \sigma_d(L_d)$, $\sigma_d(L_d)$, $G[L_{d-1}]$, $G[L_{d-1}, L_{d-2}] - \sigma_{d-1}(L_{d-1})$, $\sigma_{d-1}(L_{d-1})$, \dots in order.

- For edges in $G[L_i]$, we arbitrarily assign the usable labels to the edges.
- For edges in $G[L_i, L_{i-1}] - \sigma_d(L_i)$, by the properties of σ_i , we decompose it into edge-disjoint open trails. These trails can be classified as three types: W -type, M -type, and N -type. A trail is of W -type, or M -type, or N -type if its initial and terminal vertices are both in L_{i-1} , or both in L_i , or one in L_{i-1} and the other in L_i , respectively. The strategy of labeling each trail is to make the partial vertex sum of an internal vertex in L_{i-1} not smaller than that of an internal vertex in L_i . Assume that the usable labels of edges in $G[L_i, L_{i-1}] - \sigma_d(L_i)$ are $s, s+1, \dots, \ell$. We first label the trails of W -type and M -type. Notice that each trail of the two types have an even number of edges. Suppose that $2r$ labels are already used. To label the edges of an unlabeled trail $v_1v_2 \cdots v_a$, if it is of W -type, then assign $s+r, \ell-r, s+r+1, \ell-r-1, \dots$ to $v_1v_2, v_2v_3, v_3v_4, v_4v_5, \dots$ successively; else if it is of M -type, then assign $\ell-r, s+r, \ell-r-1, s+r+1, \dots$ to $v_1v_2, v_2v_3, v_3v_4, v_4v_5, \dots$ successively. To deal with trails of N -type, we first arrange them in pairs with possibly one trail left. Pick $T_1 = v_1v_2 \cdots v_a$ and $T_2 = u_1u_2 \cdots u_b$ a pair of trails of N -type. We may assume that $v_1 \in L_i$ and $u_1 \in L_{i-1}$. Then we assign labels $\ell-r, s+r, \ell-r-1, s+r+1, \dots, \ell-r-\frac{a}{2}+1$ to $v_1v_2, v_2v_3, v_3v_4, v_4v_5, \dots, v_{a-1}v_a$ and labels $s+r+\frac{a}{2}-1, \ell-r-\frac{a}{2}, s+r+\frac{a}{2}, \ell-r-\frac{a}{2}-1, \dots, s+r+\frac{a}{2}+\frac{b}{2}-1$ to $u_1u_2, u_2u_3, u_3u_4, u_4u_5, \dots, u_{b-1}u_b$ successively. If there is a trail of N -type left at the end, then we label it with the same strategy as labeling T_1 .
- For edges in $\sigma_i(L_i)$, observe that for each vertex $u \in L_i$, $\sigma_i(u)$ is the only unlabeled edge incident to u . As before, we define the partial vertex sum of u by

$$\phi'_f(u) := \sum_{e \sim v, e \neq \sigma_i(u)} f(e),$$

and assume $\phi'_f(u_1) \leq \phi'_f(u_2) \leq \cdots \leq \phi'_f(u_{|L_i|})$ for $u_j \in L_i$. Then assign the usable labels to $\sigma_i(u_1), \sigma_i(u_2), \dots, \sigma_i(u_{|L_i|})$ accordingly.

Finally, we verify that f is an SDDS-labeling. By the labeling rule for $\sigma_i(L_i)$'s at the end of the previous paragraph, two vertices in the same level must have distinct vertex sums. Suppose that $u \in L_i$, $v \in L_j$, $i > j$, and $\deg(u) = \deg(v) = 2d+1$. If $i \geq j+2$, then $\phi_f(u) < \phi_f(v)$ is straightforward since the label of any edge incident to u is less than that of any edge incident to v . Consider the case of $i = j+1$. Again, suppose that the labels of edges in $G[L_i, L_{i-1}] - \sigma_i(L_i)$ are $s, s+1, \dots, \ell$. It suffices to show that $\phi'_f(u) \leq d(s+\ell) \leq \phi'_f(v)$ since $\sigma_i(u) < \sigma_{i-1}(v)$. Observe that all edges incident to u are in $G[L_{i+1}, L_i]$, $G[L_i]$, or $G[L_i, L_{i-1}]$, and the edges in $G[L_{i+1}, L_i]$ or $G[L_i]$ have labels less than s . If $\deg_{G[L_i, L_{i-1}]}(u)$ is odd, then except for the edge $\sigma_i(u)$, other edges can be paired so that each pair appears successively in a trail and contributes either $s+\ell$ or $s+\ell-1$ to the vertex sum of u . Hence $\phi'_f(u)$ is at most $d(s+\ell)$. If $\deg_{G[L_i, L_{i-1}]}(u)$ is even, then except for the edge $\sigma_i(u)$, the remaining $\deg_{G[L_i, L_{i-1}]}(u) - 1$ edges can be paired so that each pair appears successively in a trail and one remaining edge is the initial or terminal edge of a trail. For each paired edges, they contribute either $s+\ell$ or $s+\ell-1$ to the vertex sum of u as before, and

for the single edge, it contributes at most ℓ to the vertex sum of u . Since $\deg(u)$ is odd, there exists one edge in $G[L_{i+1}, L_i] \cup G[L_i]$ incident to u , which has label less than s . By the discussion above, it is easily seen that $\phi'_f(u) \leq d(s + \ell)$.

The idea of showing $d(s + \ell) \leq \phi'_f(v)$ in the following is similar. All edges incident to v are in $G[L_i, L_{i-1}]$, $G[L_{i-1}]$, or $G[L_{i-1}, L_{i-2}]$, and edges in $G[L_{i-1}]$ or $G[L_{i-1}, L_{i-2}]$ have labels greater than ℓ . If $\deg_{G[L_i, L_{i-1}] - \sigma_i(L_i)}(v)$ is even, then these edges can be paired so that each pair of edges appears successively in a trail and contributes either $s + \ell$ or $s + \ell + 1$ to the vertex sum of v . Moreover, each edge in $\sigma_i(L_i)$ incident to v has label greater than ℓ . So $\phi'_f(u)$ is at least $d(s + \ell)$. Next, if $\deg_{G[L_i, L_{i-1}] - \sigma_i(L_i)}(v)$ is odd, then these edges can be paired so that each pair of edges appears successively in a trail and one remaining edge is the initial or terminal edge of a trail. For each paired edges, they contribute either $s + \ell$ or $s + \ell + 1$ to the vertex sum of v as before, and for the single edge, it contributes at least s to the vertex sum of v . Therefore, the partial vertex sum of v is at least $d(s + \ell)$. This completes the proof. \square

Next, we turn our attention to graphs which are strongly antimagic, i.e., an antimagic labeling f satisfying $\phi_f(u) < \phi_f(v)$ whenever $\deg(u) < \deg(v)$. If a graph G is strongly antimagic, then the *join* of G and a vertex v , $G \vee v$, obtained by connecting every vertex to v is also strongly antimagic by simply giving the largest $|V(G)|$ labels to edges incident to v . Moreover, spider graphs [8, 14] and double spiders graphs are strongly antimagic [3]. A simple observation by Wang and Hsiao [18] is that the strongly-antimagic property implies the k -shifted-antimagic property for all $k \geq 0$.

Proposition 2.4 [18] *If G is a strongly antimagic graph, then it is k -shifted-antimagic for all $k \geq 0$.*

Proof The proof follows from the same argument in the proof of Lemma 2.1. \square

Obviously, changing the sign of the label of each edge changes the sign of each vertex sum. By symmetry, we have the following simple fact.

Proposition 2.5 *If G is a k -shifted-antimagic graph, then it is $-(m + k + 1)$ -shifted-antimagic.*

Proof Assume that f is a k -shifted-antimagic labeling of G . Define g an injection from $E(G)$ to $\{-(k + m + 1) + 1, \dots, -(k + m + 1) + m\}$ by letting $g(e) = -f(e)$. Then $\phi_g(v) = -\phi_f(v)$ are distinct for all vertices $v \in V(G)$. Hence, G is $-(m + k + 1)$ -shifted-antimagic. \square

Does there exist an absolutely antimagic graph, i.e. a graph is k -shifted-antimagic for any integer k ? So far we have seen several examples of graphs that are k -shifted-antimagic when k is sufficiently large or sufficiently small. In addition, if G is strongly antimagic, then by Propositions 2.4 and 2.5 there are at most m possible values of k for which G is not k -shifted-antimagic. This reduces the loads of investigating whether or not a graph is absolutely antimagic. In the following, two classes of graphs are found to be absolutely antimagic. A trivial example is regular

graphs, which are shown to be antimagic in [2] and thus absolutely antimagic as no two vertices are of different degrees in a regular graph. We demonstrate absolutely antimagic graphs that are not regular.

Theorem 2.6 *Every path P_n with $n \geq 6$ is absolutely antimagic.*

Proof Claim: Every path P_n with $n \geq 3$ is strongly antimagic.

In fact, a path can be viewed as a special type of spider, which is shown strongly antimagic [14]. For the sake of completeness, we present the proof of the claim.

Proof of the Claim. We give a strongly antimagic labeling directly. Denote a path on n vertices by $P_n = v_1 v_2 \cdots v_n$. For odd n , let

$$f(v_i v_{i+1}) = \begin{cases} 1, & \text{for } i = 1; \\ i + 1, & \text{for } 2 \leq i \leq n - 2; \\ 2, & \text{for } i = n - 1. \end{cases}$$

Note that except $\phi_f(v_2) = 4$, $\phi_f(v_{n-1}) = n + 1$, and $\phi_f(v_n) = 2$, other vertex sums are distinct odd integers. For even n , let

$$f(v_i v_{i+1}) = \begin{cases} 1, & \text{for } i = 1; \\ n + 1 - i, & \text{for } 2 \leq i \leq n - 1. \end{cases}$$

Then the vertex sums are distinct odd integers except for $\phi_f(v_2) = n$ and $\phi_f(v_n) = 2$. \square

Consider P_n with $n \geq 6$. By the claim and Proposition 2.4, P_n is k -shifted-antimagic for any $k \geq 0$. Also, it is k -shifted-antimagic for $k \leq -n$ by Proposition 2.5. To prove that P_n is k -shifted-antimagic for $-n + 1 \leq k \leq -1$, it is sufficient to verify those k 's with $-n/2 \leq k \leq -1$ by the symmetry property in Proposition 2.5. For each k in this range, we define a labeling f_k as follows. For $-n/2 \leq k \leq -3$ or $k = -1$, let $f_k(v_{|k|} v_{|k|+1}) = 0$. Now consider the two subpaths $Q_1 = v_1 v_2 \cdots v_{|k|}$ and $Q_2 = v_{|k|+1} \cdots v_{n-1} v_n$. By the same labeling method in the claim, we use $1, 2, \dots, n - |k| - 1$ to label $E(Q_2)$ while using $1, 2, \dots, |k| - 1$ to label $E(Q_1)$ but adding a negative sign to the label of each edge in $E(Q_1)$. Then the vertex sums at vertices in $V(Q_i)$ for each i are distinct by the claim. Moreover, the vertex sums are all negative at $V(Q_1)$ and positive at $V(Q_2)$.

The last case is $k = -2$. Then for odd n , let

$$f_{-2}(v_i v_{i+1}) = \begin{cases} -1, & \text{for } i = 1; \\ 1, & \text{for } i = 2; \\ 0, & \text{for } i = 3; \\ i - 2, & \text{for } i \geq 4. \end{cases}$$

We have vertex sums $-1, 0, 1, 2, 5, 7, 9, \dots, 2n - 7, n - 3$ at v_1, \dots, v_n , respectively. For even n , define

$$f_{-2}(v_i v_{i+1}) = \begin{cases} 0, & \text{for } i = 1; \\ -1, & \text{for } i = 2; \\ n - i, & \text{for } i \geq 3. \end{cases}$$

The vertex sums are $0, -1, n - 4, 2n - 7, 2n - 9, \dots, 3, 1$ at v_1, \dots, v_n , respectively. The proof is completed. \square

A path P_n is not k -shifted-antimagic for any integer k when $n = 2$, while it is k -shifted-antimagic for any integer k if $n \geq 6$. For $n \in \{3, 4, 5\}$, P_n is not k -shifted-antimagic for some integers k . In Sect. 3, we will present several classes of graphs, including P_3 , P_4 , and P_5 , which are not k -shifted-antimagic for some k .

3 Graphs that Are Not k -Shifted-Antimagic

This section focuses on graphs that are not k -shifted-antimagic. We first show some connected graphs, specifically trees, that are not k -shifted-antimagic with certain values of k . These cases of trees shall be classified according to diameter. Then for some examples of disconnected graphs we characterize the sufficient and necessary conditions of the value k for which they are k -shifted-antimagic. Despite the fact that every connected graph G might be antimagic, we prove that there exists a disconnected graph G' containing G as a component but G' is not antimagic.

3.1 Trees with Diameter at Most Four

The diameter of a tree G is the largest integer d so that G contains a path P_{d+1} as a subgraph. The path P_2 is the only tree of diameter one, which is not k -shifted antimagic for any k . A tree of diameter two is also called a star S_n , where $n \geq 2$ is the number of leaves.

Proposition 3.1 *For $n \geq 2$, the star S_n is k -shifted-antimagic if and only if $k \notin \{-\frac{1}{2}n - 1, -\frac{1}{2}n\}$ for even n and $k \neq -\frac{1}{2}(n + 1)$ for odd n .*

Proof Let f be an injection from $E(S_n)$ to $\{k + 1, \dots, k + n\}$. Then we have vertex sums $\phi_f(v) = k + 1, \dots, k + n$ for the leaves, and $\phi_f(v) = \sum_{i=1}^n (k + i) = nk + \frac{1}{2}(n^2 + n)$ for the unique internal vertex. Thus, f is not k -shifted-antimagic if and only if

$$k + 1 \leq nk + \frac{1}{2}(n^2 + n) \leq k + n.$$

Solving the inequalities yields $-\frac{1}{2}n - 1 \leq k \leq -\frac{1}{2}n$. \square

A tree of diameter three must contain a path P_4 with all remaining vertices adjacent to one of the two internal vertices of P_4 . We call it a *double star* $S_{a,b}$, where $a, b \geq 1$ are the numbers of leaves adjacent to the two internal vertices, respectively. Without loss of generality, assume $a \geq b$ in $S_{a,b}$.

Theorem 3.2 A double star $S_{a,b}$ is absolutely antimagic if and only if $(a,b) \notin \{(2,1)\} \cup \{(a,1) \mid a \text{ is odd}\}$. Moreover, $S_{2,1}$ is k -shifted-antimagic if and only if $k \notin \{-2, -3\}$, and $S_{a,1}$ is k -shifted-antimagic if and only if $k \neq -\frac{1}{2}(a+3)$.

Proof Denote the vertices of $S_{a,b}$ by v, u, v_1, \dots, v_a , and u_1, \dots, u_b such that v and u are the internal vertices, and v_i 's and u_j 's are the leaves adjacent to v and u , respectively. Fix an integer k and let n_+ and n_- be the number of positive and negative integers in $\{k+1, k+2, \dots, k+a+b+1\}$, respectively. By symmetry, we assume $n^+ \geq n^-$ in the sequel (Fig. 1).

• Case 1: $b \geq 2$.

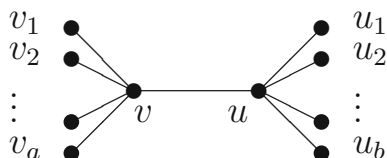
If $n_- = 0$, then we assign $k+a+b+1, k+a+b, \dots, k+1$ to edges $vu, v_1v, u_1u, v_2v, u_2u, \dots, v_bv, u_bu$, possibly $v_{b+1}v, v_{b+2}v, \dots$ in order. Since the largest label is assigned to vu and $f(v_iv) > f(u_iu)$ for all $1 \leq i \leq b$, we have $\phi_f(v) > \phi_f(u) > \phi_f(w)$ for any pendent vertex w . Now we suppose that both n_+ and n_- are nonzero.

When $n_+ - n_- \geq 2$, first assign $\pm j$, for $1 \leq j \leq n_-$, to a pair of edges u_iu 's or a pair of edges v_iv 's simultaneously. Let H be the subgraph whose edge set is the set of all unlabeled edges. Observe that H must be either a star or a double star. If H is a star, then we assign the remaining labels to the edges of H arbitrarily. Then one of the vertex sums $\phi_f(u)$ and $\phi_f(v)$ is equal to $f(vu)$ and the other is equal to $(n_- + 1) + \dots + n_+ > n_+$. The vertex sums of other pendent vertices are just the labels of the pendent edges. None of them is equal to $f(vu)$ and all are at most n_+ . If H is a double star, then we label the edges of H as the case of $n_- = 0$. Without loss of generality, assume $d_H(v) \geq d_H(u)$. First assign n_+ to vu , then assign $n_+ - 1, n_+ - 2, \dots$ to an edge v_iv and an edge u_iu alternatively. We have $\phi_f(v) > \phi_f(u) \geq 2n_+ - 2 > \phi_f(w)$ for any pendent vertex w .

When $n_+ - n_- = 1$, note that since $n_+ + n_- + 1 = |E(S_{a,b})| = a + b + 1$ and $a \geq b$, we have $n_- \geq b \geq 2$ and the parities of a and b are distinct. First, we assign 0 to vu and $\pm j$ simultaneously to a pair of edges u_iu 's or a pair of edges v_iv 's for $1 \leq j \leq n_- - 2$, then leave three unlabeled edges incident one internal vertex and two unlabeled edges incident to the other one. Next, assign n_+, n_- , and $n_- - 1$ to the three unlabeled edges incident to the same vertex, and $-n_-$, and $-(n_- - 1)$ to the remaining edges. We have $\{\phi_f(u), \phi_f(v)\} = \{3n_+ - 3, -(2n_- - 1)\}$, so the vertex sums of u and v are different from that of any pendent vertex.

When $n_+ = n_-$, we assign 0 to vu and $-n_-, -(n_- - 1), \dots, -1, 1, 2, \dots, n_+$ to

Fig. 1 The double star $S_{a,b}$



the edges $u_1u, u_2u, \dots, u_bu, v_1v, v_2v, \dots, v_av$ accordingly. Note that $b \geq 2$ implies $\phi_f(u) \leq -n_- - (n_- - 1)$ and $\phi_f(v) \geq n_+ + (n_+ - 1)$. Then labeling f gives $\phi_f(u) < \phi_f(w) < \phi_f(v)$ for any pendent vertex w .

- Case 2: $b = 1$.

Let us first assume $a \notin \{1, 2\}$. If $n_- = 0$, we then assign $k + a + 2, k + a + 1, \dots, k + 1$ to the edges $vu, v_1v, u_1u, v_2v, v_3v, \dots, v_av$ in order. By the argument in Case 1, this is a k -shifted-antimagic labeling for $S_{a,1}$.

When $n_+ - n_- \geq 2$, we assign n_+ to vu , 0 to u_1u , and other labels to the remaining edges. This labeling gives $\phi_f(v) > \phi_f(u) = n_+ > \phi_f(w)$ for any pendent vertex w .

When $n_+ - n_- = 1$, since $n_+ + n_- + 1 = a + 2$, this implies a is even and $n_- \geq 2$. Then assign $-(n_- - 1)$ to vu , $-n_-$ to u_1u , and other labels to the remaining edges. We have $\phi_f(v) = n_+ + n_- > \phi_f(w) > -(2n_- - 1) = \phi_f(u)$ for any pendent vertex w .

When $n_+ = n_-$, a is odd and $k = -\frac{1}{2}(a + 3)$. We show that $S_{a,1}$ is not k -shifted-antimagic. It is easy to see that 0 cannot be assigned to the internal edge vu or the edge u_1u , because the former leads to $\phi_f(u) = \phi_f(u_1) = f(u_1u)$ and the latter leads to $\phi_f(v) = \phi_f(u_1) = 0$. Thus, 0 only goes to some pendent edge v_iv . Now that $f(u_1u) = x \neq 0$, if $f(vu) = -x$, then $\phi_f(u) = \phi_f(v_1) = 0$ which is forbidden. Else, we have $f(v_iv) = -x$ for some $j \neq i$, and then the labels of edges incident to v , except for vv_j , sum to zero. Therefore $\phi_f(v) = \phi_f(v_j) = -x$, which is also a contradiction. Consequently, $S_{a,1}$ is not k -shifted-antimagic for $k = -\frac{1}{2}(a + 3)$.

Here comes the discussion of k -shifted-antimagic labelings of $S_{1,1}$ (or equivalently P_4) and $S_{2,1}$. For $S_{1,1}$, setting $f(v_1v) = k + 1, f(vu) = k + 3$, and $f(u_1u) = k + 2$ for $k \geq -1$ yields a k -shifted-antimagic labeling. By Proposition 2.5, $S_{1,1}$ is k -shifted-antimagic for $k \leq -3$. When $k = -2$, observe that 0 cannot be assigned to the internal edge vu and the ± 1 cannot be assigned to two incident edges, thereby (-2) -shifted-antimagic labelings do not exist. For $S_{2,1}$, setting $f(v_1v) = k + 2, f(v_2v) = k + 3, f(vu) = k + 4$, and $f(u_1u) = k + 1$ for $k \geq -1$ obtains a k -shifted-antimagic labeling, and again $S_{2,1}$ is k -shifted-antimagic for $k \leq -4$ by Proposition 2.5. When $k = -2$ (the argument of $k = -3$ is similar by symmetry), it is easily seen that 0 cannot be assigned to the internal edge vu . If 0 is assigned to uu_1 , then the two vertices incident to the edges labeled by 2 have the same vertex sum. Thus, 0 can only be assigned to either v_1v or v_2v . In this case, there is either another vertex whose vertex sum equals 0 or a pair of vertices whose vertex sums are equal to 1, no matter how we label the remaining three edges. According to the discussion above, no k -shifted-antimagic labeling exists for $k \in \{-2, -3\}$. \square

For trees of diameter four, we have the following partial results.

Theorem 3.3 *The path P_5 is k -shifted-antimagic if and only if $k \notin \{-2, -3\}$.*

Proof Denote P_5 by $v_1v_2v_3v_4v_5$. By the claim in Theorem 2.6 and Proposition 2.5, P_5 is k -shifted-antimagic for $k \geq 0$ and $k \leq -5$.

To show that P_5 is (-1) -shifted-antimagic, we label $v_1v_2, v_2v_3, v_3v_4, v_4v_5$, with 0, 1, 3, 2, respectively. It follows by symmetry that P_5 is (-4) -shifted-antimagic.

It suffices to show that P_5 is not k -shifted-antimagic for $k = -2$ ($k = -3$ follows by symmetry). Assume that f is a (-2) -shifted-antimagic labeling of P_5 . Then $\{f(v_i v_{i+1}) \mid 1 \leq i \leq 4\} = \{-1, 0, 1, 2\}$. If $f(v_2 v_3) = 0$, then $\phi_f(v_1) = \phi_f(v_2)$, which is not allowed. By symmetry, $f(v_3 v_4) \neq 0$ and we may assume $f(v_1 v_2) = 0$. If -1 and 1 are labeled to two incident edges, then the common vertex of the two edges has a vertex sum 0, which is the same as $\phi_f(v_1)$, a contradiction. The remaining possible labelings are $f(v_1 v_2) = 0, f(v_2 v_3) = 1, f(v_3 v_4) = 2, f(v_4 v_5) = -1$, or $f(v_1 v_2) = 0, f(v_2 v_3) = -1, f(v_3 v_4) = 2, f(v_4 v_5) = 1$. In either case, we have two vertices whose vertex sums are equal to one. Thus, the labeling f does not exist. \square

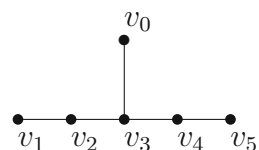
Define P'_5 to be the graph obtained by attaching an edge to the central vertex of P_5 , as illustrated in Fig. 2.

Theorem 3.4 *The graph P'_5 is k -shifted-antimagic if and only if $k \neq -3$.*

Proof We first label $v_1 v_2, v_2 v_3, v_3 v_4, v_4 v_5$, and $v_0 v_3$ with 2, 4, 5, 3, and 1, respectively. This is a strongly antimagic labeling of P'_5 . So the remaining cases are $k \in \{-1, -2, -3, -4, -5\}$. A (-1) -shifted-antimagic labeling for P'_5 can be constructed by extending an antimagic labeling of P_5 to P'_5 by giving 0 to the edge $v_0 v_3$. Next, we can label $v_1 v_2, v_2 v_3, v_3 v_4, v_4 v_5$, and $v_0 v_3$ with 3, 2, 1, 0, and -1 , respectively, to get a (-2) -shifted-antimagic labeling. The existence of k -shifted-antimagic labeling for $k \in \{-4, -5\}$ follows by Proposition 2.5 as before.

We now show that P'_5 is not (-3) -shifted-antimagic. Clearly, when labeling the edges with $-2, -1, 0, 1, 2$, the number 0 can only be labeled to $v_1 v_2, v_4 v_5$, or $v_0 v_3$. When assigning 0 to $v_1 v_2$ (the case of $v_4 v_5$ is similar), the labels of $v_2 v_3, v_3 v_4$, and $v_0 v_3$ must contain a pair of labels $\pm x$. If this pair happens at $v_2 v_3$ and $v_3 v_4$, then the vertex sums of v_0 and v_3 are equal; else if the pair happens at $v_3 v_4$ and $v_0 v_3$, then the vertex sums of v_2 and v_3 are equal; else if the pair happens at $v_2 v_3$ and $v_0 v_3$, then the vertex sums of v_4 is zero, since the labels of $v_3 v_4$ and $v_4 v_5$ must be different by a negative sign, which is equal to that of v_1 . Therefore, 0 only goes to $v_0 v_3$. Since the vertex sum of v_0 is zero, it is forbidden to assign two labels x and $-x$ to two incident edges $v_i v_{i+1}$ and $v_{i+1} v_{i+2}$ for $1 \leq i \leq 3$. We may assume the labels of $v_1 v_2, v_2 v_3, v_3 v_4$, and $v_4 v_5$ are $x, y, -x$, and $-y$, respectively, and $x \neq \pm y$. Then the vertex sums of v_1, \dots, v_5 are $x, x+y, y-x, -x-y, -y$, respectively. Notice that $\{\pm x, \pm y\} = \{\pm 1, \pm 2\}$, so one of the equations $x = 2y, x = -2y, y = 2x$, and $y = -2x$ must hold. However, any one of the equations leads to a pair of equal numbers in $\{x, x+y, y-x, -x-y, -y\}$. Consequently, no (-3) -shifted-antimagic labeling exists for P'_5 . \square

Fig. 2 The graph P'_5



3.2 Disconnected Graphs

The study of antimagic labelings on disconnected graphs does not draw as much attention as connected graphs. In contrast to the rare examples of non-absolutely antimagic connected graphs, there are abundant examples of disconnected graphs that are k -shifted-antimagic with restricted values k . Moreover, the non-absolutely antimagic connected graphs we have seen in Sect. 3.1 are not k -shifted-antimagic only for one or two specific k . In this subsection, we also demonstrate graphs that are not k -shifted-antimagic graphs for many k 's.

First, we introduce some non-0-shifted-antimagic (i.e. not antimagic) disconnected graphs. Let $G_1 + G_2 + \cdots + G_c$ be a graph consisting of c connected components G_1, \dots, G_c . Denote it as cG if all components are isomorphic to G . Shang, Lin, and Liaw [16] prove that if $G = cP_3 + S_n$ is antimagic if and only if $c \leq \min\{2n + 1, \frac{2n-5+\sqrt{8n^2-24n+17}}{2}\}$. In [14], Shang pointed out that cP_3 is antimagic if and only if $c = 1$. In the following, we show that if a graph contains too many components isomorphic to P_3 , then it is not antimagic.

Theorem 3.5 *For any graph G , there exists a constant $c = c(G)$ such that the graph $G + cP_3$ is not antimagic.*

Proof Given a graph $G = (V, E)$ with $|V(G)| = n$ and $|E(G)| = m$, the graph $G + cP_3$ contains $n + 3c$ vertices and $m + 2c$ edges. Denote the c paths by $u_1v_1w_1, u_2v_2w_2, \dots, u_cv_cw_c$. Assume that f is an antimagic labeling of $G + cP_3$. Then we have $\phi_f(u_i) = f(u_iv_i)$ and $\phi_f(w_i) = f(v_iw_i)$ for $1 \leq i \leq c$, which are all distinct. Thus, either $\phi_f(v_i) \in \{1, 2, \dots, m + 2c\} - \{f(u_jv_j), f(v_jw_j) \mid 1 \leq j \leq c\}$, or $\phi_f(v_i) > m + 2c$. When $c > m$, $\phi_f(v_i) > m + 2c$ holds for at least $c - m$ v_i 's. Note that $\phi_f(v_i) = \phi_f(u_i) + \phi_f(w_i)$. Thus, the total vertex sum of these v_i 's with $\phi_f(v_i) > m + 2c$ and their neighbors u_i 's and w_i 's is at least $2[(m + 2c + 1) + (m + 2c + 2) + \cdots + (m + 2c + (c - m))] = (1 + m + 5c)(c - m)$. Observe that the total vertex sum of all vertices is $\sum_{v \in V(G)} \phi_f(v) = 2 \sum_{e \in E(G)} f(e) = (1 + m + 2c)(m + 2c)$. We then compare it with the weight sum of the vertices on the P_3 's. Observe that

$$\begin{aligned} (1 + m + 2c)(m + 2c) &= 1 + m + m^2 + 2c + 4mc + 4c^2 \\ &< -m - m^2 + (1 - 4m)c + 5c^2 \\ &= (1 + m + 5c)(c - m) \end{aligned}$$

for sufficiently large c . This leads to a contradiction. Consequently, $G + cP_3$ is not antimagic for sufficiently large c . \square

The next result points out that cP_3 is an example of the k -shifted-antimagic graphs for all but finitely many k . In addition, the number of values of k for which cP_3 is not k -shifted-antimagic increases as c increases, and they form a set of consecutive integers.

Theorem 3.6 *The graph cP_3 is k -shifted-antimagic if and only if*

$$k \notin \{-\lfloor \frac{5c}{2} \rfloor, -\lfloor \frac{5c}{2} \rfloor + 1, \dots, \lfloor \frac{c}{2} \rfloor - 1\}.$$

Proof We show that cP_3 has a k -shifted-antimagic labeling if and only if $k \geq \lfloor \frac{c}{2} \rfloor$ or $k \leq -(\lfloor \frac{5c}{2} \rfloor + 1)$. As in the previous theorem, let the c copies of P_3 's be $u_1v_1w_1, \dots, u_cv_cw_c$.

We first prove that there exists a k -shifted-antimagic labeling of cP_3 for $k = \lfloor \frac{c}{2} \rfloor$. If $c = 2\ell + 1$, then all the labels are elements in $\{\ell + 1, \dots, 5\ell + 2\}$. Assign each of the following pairs of labels to the edges in one P_3 : $\{\ell + 1, 4\ell + 2\}, \{\ell + 3, 4\ell + 1\}, \dots, \{\ell + 2i - 1, 4\ell + 3 - i\}, \dots, \{3\ell + 1, 3\ell + 2\}$, and $\{\ell + 2, 5\ell + 2\}, \{\ell + 4, 5\ell + 1\}, \dots, \{\ell + 2i, 5\ell + 3 - i\}, \dots, \{3\ell, 4\ell + 3\}$. Note that the vertex sums of v_i 's are $5\ell + 3, 5\ell + 4, \dots, 7\ell + 3$, which are all distinct and greater than $5\ell + 1$. So this is a $\lfloor \frac{c}{2} \rfloor$ -shifted-antimagic labeling of cP_3 for odd c . If $c = 2\ell$, then we give each of the following pairs of labels to the edges in one P_3 : $\{\ell + 1, 4\ell\}, \{\ell + 3, 4\ell - 1\}, \dots, \{\ell + 2i - 1, 4\ell + 1 - i\}, \dots, \{3\ell - 1, 3\ell + 1\}$, and $\{\ell + 2, 5\ell\}, \{\ell + 4, 5\ell - 1\}, \dots, \{\ell + 2i, 5\ell + 1 - i\}, \dots, \{3\ell, 4\ell + 1\}$. Note that the vertex sums of v_i 's are $5\ell + 1, 5\ell + 2, \dots, 6\ell$ and $6\ell + 2, 6\ell + 3, \dots, 7\ell + 1$, which are all distinct and greater than $5\ell + 2$. Again, this is a $\lfloor \frac{c}{2} \rfloor$ -shifted-antimagic labeling of cP_3 for even c . Observe that the above labelings are indeed a strongly antimagic labelings. Hence there exists a k -antimagic labeling for every $k \geq \lfloor \frac{c}{2} \rfloor$.

On the other hand, assume that cP_3 is k -shifted-antimagic for some $k \geq 0$, and let f be such a labeling. Since each edge is incident to a vertex of degree one, the vertex sums of the vertices of degree one are exactly the labels of the edges, namely $k + 1, k + 2, \dots, k + 2c$. Hence, we have $\phi_f(v_i) > k + 2c$ for all i . Without loss of generality, assume $\phi_f(v_1) < \phi_f(v_2) < \dots < \phi_f(v_c)$, and $\phi_f(v_1) > k + 2c$. We have

$$\sum_{i=1}^c (k + 2c + i) \leq \sum_{i=1}^c \phi_f(v_i) = \sum_{e \in E(cP_3)} f(e) = \sum_{i=1}^{2c} (k + i).$$

Simplifying the above inequality leads to $k \geq \frac{c-1}{2}$, which is essentially the same as $k \geq \lfloor \frac{c}{2} \rfloor$. By Proposition 2.5, when k is negative, cP_3 is k -shifted-antimagic if and only if $-(m + k + 1) \geq \lfloor \frac{c}{2} \rfloor$. Equivalently, $k \leq -m - 1 - \lfloor \frac{c}{2} \rfloor = -\lfloor \frac{5c}{2} \rfloor - 1$. \square

A $\{P_{k_1}, \dots, P_{k_s}\}$ -free linear forest is a disjoint union of paths such that none of the paths is P_{k_i} for any i . It is conjectured by Shang [15] that every $\{P_2, P_3\}$ -free linear forest is antimagic. The following result demonstrates an example of a $\{P_2, P_3\}$ -free linear forest that is not k -shifted-antimagic with nonconsecutive k 's.

Theorem 3.7 *The graph $2P_4$ is k -shifted-antimagic if and only if $k \notin \{-2, -5\}$.*

Proof By Proposition 2.5, it suffices to verify that $G = 2P_4$ is k -shifted-antimagic for $k = -3$ and $k \geq -1$, but not (-2) -shifted-antimagic. Denote the two paths by $v_1v_2v_3v_4$ and $u_1u_2u_3u_4$.

We first show that it is not (-2) -shifted-antimagic. Suppose that f is a (-2) -shifted-antimagic labeling with labels $-1, 0, 1, 2, 3$, and 4 . If $f(v_2v_3) = 0$, then not

only the vertex sums $\phi_f(v_1) = \phi_f(v_2)$ but also $\phi_f(v_3) = \phi_f(v_4)$. Hence $f(v_2v_3) \neq 0$. Similarly, $f(u_2u_3) \neq 0$. Without loss of generality, we may assume $f(v_1v_2) = 0$.

- Case 1: $f(v_2v_3) = -1$.
Let $f(v_3v_4) = x$. If $x = 1$, then $\phi_f(v_3) = \phi_f(v_1) = 0$. If $x = 2$, then $\phi_f(v_3) = 1$, and 1 must be labeled to u_2u_3 . Thus, $\{f(u_1u_2), f(u_3u_4)\} = \{3, 4\}$. Consequently, there exist an internal vertex and a leaf having the same vertex sum in this path. If $x = 3$, then $\{\phi_f(v_3), \phi_f(v_4)\} = \{2, 3\}$. To avoid the same vertex sum, neither can we label two incident edges with 1 and 2, nor can we label a pendent edge with 2 in the path $u_1u_2u_3u_4$, so $x \neq 3$. The last possibility is $x = 4$. Then we have $\{\phi_f(v_3), \phi_f(v_4)\} = \{3, 4\}$. The edges in the other path $u_1u_2u_3u_4$ labeled by 1, 2, and 3 must induce the vertex sum 3 or 4 at some vertex, which is not allowed. As a consequence, $x \neq 4$.
- Case 2: $f(v_3v_4) = -1$.
The argument is exactly the same as the argument above.
- Case 3: $f(u_1u_2) = -1$ (or $f(u_3u_4) = -1$).
Let $f(u_2u_3) = x$. If $x = 1$, then $\phi_f(u_2) = \phi_f(v_1) = 0$. When $x \geq 2$, one of the vertices v_2, v_4, u_4 must have vertex sum $x - 1$. So we exclude this case.
- Case 4: $f(u_2u_3) = -1$.
No matter how the remaining edges are labeled, $\{\phi_f(v_2), \phi_f(v_4), \phi_f(u_1), \phi_f(u_4)\} = \{1, 2, 3, 4\}$ and $\{\phi_f(u_2), \phi_f(u_3)\} \subset \{0, 1, 2, 3\}$. Thus, $\{\phi_f(v_2), \phi_f(v_4), \phi_f(u_1), \phi_f(u_4)\} \cap \{\phi_f(u_2), \phi_f(u_3)\} \neq \emptyset$.

So, f cannot be a (-2) -shifted-antimagic labeling.

For $k = -1$, we label the edges as follows: $f(v_1v_2) = 0$, $f(v_2v_3) = 4$, $f(v_3v_4) = 1$, $f(u_1u_2) = 2$, $f(u_2u_3) = 5$, $f(u_3u_4) = 3$. Observe that f is a strongly antimagic labeling. Hence $2P_4$ is k -shifted-antimagic for all $k \geq -1$.

Finally, for $k = -3$, we label v_1v_2 , v_2v_3 , and v_3v_4 , with -2 , -1 , and 0 , respectively, and also label u_1u_2 , u_2u_3 , and u_3u_4 with 2 , 3 , and 1 , respectively. It is easy to verify that all vertex sums are all distinct. The proof is complete. \square

In addition to $2P_4$, we have another graph which is not k -shifted-antimagic with nonconsecutive k 's.

Theorem 3.8 *The graph $2S_3$ is k -shifted-antimagic if and only if $k \notin \{-2, -5\}$.*

Proof Denote the vertices of one star by v, v_1, v_2, v_3 and those of the other star by u, u_1, u_2, u_3 , where v_i 's and u_i 's are the leaves. Let us label the edges v_1v, v_2v, v_3v with $k + 1, k + 3, k + 6$, and the edges u_1u, u_2u, u_3u with $k + 2, k + 4, k + 5$, respectively. Then $\phi_f(u) > \phi_f(v) = 3k + 10 > k + 6$ if and only if $k > -2$. Hence $2S_3$ is k -shifted-antimagic for all $k > -2$, and by Proposition 2.5, for all $k < -5$. For $k = -3$, we assign $-2, -1, 0$ to the edges of one star and $1, 2, 3$ to the edges of the other star. This is clearly a (-3) -shifted-antimagic labeling of $2S_3$. Again, by Proposition 2.5, $2S_3$ is (-4) -shifted-antimagic. Finally, we show that $2S_3$ is not k -shifted-antimagic for $k \in \{-2, -5\}$. We only show the case $k = -2$. When $k = -2$, the usable labels are $-1, 0, 1, 2, 3, 4$. For any labeling f , we have $0 \leq \phi_f(v) \leq 9$,

$0 \leq \phi_f(u) \leq 9$, and $\phi_f(v) + \phi_f(u) = 9$. Therefore, at least one of $\phi_f(v)$ and $\phi_f(u)$ is less than 5, which is equal to the vertex sum of some leaf. So $2S_3$ is not (-2) -shifted-antimagic. \square

4 Concluding Remarks and Future Work

We first want to stress that the “level-by-level” algorithm mentioned in Sect. 2 could be applicable to graphs containing vertices of even degree with some extra conditions. For example, if all vertices of even degree have distinct degrees, then we can prove the existence of an SDDS-labeling of such a graph using the same argument in the proof of Theorem 2.3. Alternatively, we may assume that all vertices of even degree have the same degree, and then apply a minimum d -covering pair method by Chang et al. [2] used to find the antimagic labeling of even regular graphs. Readers may consult [2] for more details. Unfortunately, we are not able to generalize their method to non-regular graphs. Despite the fact that the existence of the k -shifted antimagic labeling for graphs with mixed even degrees is still unknown, we believe that all graphs are k -shifted-antimagic for some k and pose the following conjecture:

Conjecture 4.1 *Every graph is k -shifted-antimagic for $|k|$ sufficiently large if it does not contain a component isomorphic to P_2 .*

Theorem 3.5 states that for any graph G , the graph $G + cP_3$ is not antimagic for sufficiently large c . In other words, the union of antimagic graphs could be not antimagic. We are curious that does there exist any other type of graphs $G = G_1 + G_2 + \cdots + G_k$, not many $G_i = P_3$, such that each component G_i is antimagic but not G ?

If a counterexample exists for Conjecture 4.1, then the graph must contain a component which is not absolutely antimagic. The reason is that if each G_i is absolutely antimagic, then G is k -shifted-antimagic for sufficiently large $|k|$. To see this, we first find an antimagic labeling of G_1 and a k -shifted-antimagic labeling of G_i with $k = \sum_{j=1}^{i-1} |E(G_j)|$ for $i \geq 2$. These labelings together form an SDDS-labeling of G , and, by Lemma 2.1, G is k -shifted for sufficiently large k . Thus, a more fundamental question is whether or not a connected graph is absolutely antimagic.

In Sect. 3.1, some trees which are not absolutely antimagic were presented. We inspected some small examples of trees of diameter at least five and graphs containing a cycle, but it turns out that all those small graphs are absolutely antimagic. It may be interesting to answer the following two questions.

Question 4.1 Find a tree T of diameter at least five which is not k -shifted-antimagic for some integer k .

Question 4.2 Find a connected graph G containing a cycle, which is not k -shifted-antimagic for some integer k .

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